

# ON MODEL COMPLETION OF $T_{\text{aut}}$

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Partially supported by the Israel Science Foundation. Publication E34.  
I would like to thank Alice Leonhardt for the beautiful typing.

Typeset by  $\mathcal{A}\mathcal{M}\mathcal{S}$ - $\text{\textsf{TeX}}$

## ANNOTATED CONTENT

## §0 Introduction

## §1

[We characterize stable  $T$  for which the model completion of  $T_{\text{aut}}$  is stable (i.e., every completion is).]

## §2

[We prove that “some completion is stable” is different and characterize it.]

## §3

[We prove that if  $T$  is stable,  $T_{\text{aut}}$  has a model completion,  $T_*$  is an unstable complete of  $T_{\text{aut}}^{\text{mc}}$ , then  $T_*$  satisfies NSOP<sub>3</sub>. Moreover, simplicity is preserved.]

## §0 INTRODUCTION

On the subject, history and background see [BlSh 759]. For a complete first order  $T$  they dealt with the existence of the model completion  $T_{\text{aut}}$  of  $T \cup \{\sigma \text{ is an automorphism (for } \tau_T)\}$ .

We may ask:

0.1 Question: If  $T$  is stable and  $T_{\text{aut}}$  has model completion  $T_{\text{aut}}^{\text{mc}}$ , when is (every) completion of  $T_{\text{aut}}^{\text{mc}}$  stable?

We answer in 1.6 (observation 1.7 deals with some obvious things).

Section 1 raises some question which we discuss below (assuming  $T$  stable,  $T_{\text{aut}}^{\text{mc}}$  exists) some of which are answered below.

0.2 Question: 1) Can we in Claim 1.6 below replace “every completion of  $T_{\text{aut}}^{\text{mc}}$  is stable” by “some completion of  $T_{\text{aut}}^{\text{mc}}$  is stable”?

2) The “unstable” in 1.6 clause (a) can be replaced by “having the independence property”; but can  $T_{\text{aut}}^{\text{mc}}$  be completed to a theory with the strict order property? The  $\text{SOP}_n$ ’s?

3) What occurs if  $T_{\text{aut}}^{\text{mc}}$  does not exist, can we still say something?

4) Point out that (a)( $\equiv$  (b)) of 1.6 holds (for some stable  $T$  for which  $T_{\text{aut}}^{\text{mc}}$  exists) and fails for others.

5) Show for stable  $T$  with  $T_{\text{aut}}^{\text{mc}}$ , that no completion  $T_*$  of  $T_{\text{ut}}^{\text{dc}}$  has the explicit ncp (which means that for some first order  $E(\bar{x}, \bar{y}, \bar{z})$ , for every  $n$  for some  $\bar{c} \subseteq \mathfrak{C}$ ,  $E(\bar{x}, \bar{y}, \bar{c})$  is an equivalent relation which has  $\geq n, < \aleph_0$  equivalence classes); a stronger version is

6) For such  $T, T_*$  can  $T_*$  have obstructions (see §4)?

7) What if we use  $\sigma_1, \sigma_2$ ? What about  $\sigma_1, \dots, \sigma_n$ ? What about pairwise commuting  $\sigma_1, \dots, \sigma_n$ ? This is like  $(T_{\text{aut}})_{\text{aut}}$  for  $n = 2$ .

8) Is there unstable  $T$  such that  $T_{\text{aut}}$  has model completion? (A conjecture stating that had been the starting point of Kikyo Shelah [KkSh 748]).

0.3 Discussion: We prove that:

- (A) on 0.2(1), for some  $T$  (stable with  $T_{\text{aut}}^{\text{mc}}$  existing), some completion of  $T_{\text{aut}}^{\text{mc}}$  are stable and some are not (still we may wonder on a general characterization, see 2.7 below)
- (B) we shall show that for no such  $T$  is any completion of  $T_{\text{aut}}^{\text{mc}}$  with the strict order property and even have  $\text{NSOP}_3$ , see 3.1
- (C) we can look at the class of existentially closed models of  $T_{\text{aut}}$  (see [ShUs 789] and references there); the results are similar.

Note

0.4 *Observation.* [Here?]

- ( $\alpha$ ) for  $T =$  theory of equality,  $T_{\text{aut}}$  has a model completion and all completion of  $T_{\text{aut}}^{\text{mc}}$  are stable
- ( $\beta$ ) for  $T$  from 2.1, some completions of  $T_{\text{aut}}^{\text{mc}}$  are stable and some are not
- ( $\gamma$ ) for  $T = \text{Th}(M \upharpoonright \{E, F_1, F_2, Q\})$ ,  $M$  from 2.1, we get that all the completions of  $T_{\text{aut}}^{\text{mc}}$  are unstable.

I think

0.5 Quesiton: What about getting (in §3) that

- (a)  $T_{\text{aut}}^{\text{mc}}$  is simple in §3?
- (b) even if  $T$  is just simple,  $T_{\text{aut}}^{\text{mc}} \models \text{NSOP}_3$
- (c) non elementary class (true).

See below.

§1 ON THE STABILITY OF MODEL COMPLETION  
FOR  $T_{\text{aut}}$  ( $= T + \sigma$  AN AUTOMORPHISM)

**1.1 Hypothesis.** 1)  $T$  is first order complete and for notational simplicity every formula is equivalent to a relation and  $\tau_T$  having only predicates.  
2)  $\mathfrak{C}$  is the monster model of  $T$ .

**1.2 Definition.** 1)  $T_{\text{aut}}$  is  $T \cup \{\sigma \text{ is an automorphism (for } \tau_T)\}$ , so  $\sigma$  is a new unary function symbol that is  $T_{\text{aut}} = T \cup \{(\forall x_0, \dots, x_{n-1})[R(x_0, \dots, x_{n-1}) \equiv R(\sigma(x_0), \dots, \sigma(x_{n-1}))]\} : R \text{ an } n\text{-place predicate of } \tau_T\}$ .

2)  $T_{\text{aut}}^{\text{mc}}$  is the model completion, if it exists.

3) Let  $T_*$  denote any completion of  $T_{\text{aut}}^{\text{mc}}$  and  $\sigma_*$  or  $\sigma^{N^+}$  is an automorphism.

4) A completion  $T_*$  of  $T_{\text{aut}}^{\text{mc}}$  is cute if it has a model  $N^+$  such that for some  $M^+ \subseteq N^+$  we have  $\sigma^{N^+} = \text{id}_{N^+}$ .

**1.3 Definition.** For  $T$  as in 0.2 let:

1)  $K_{\text{aut}}(T) =$  the class of models of  $T_{\text{aut}}$ .

2)  $K_{\text{aut}}^{\text{ec}}(T) =$  the class of e.c. models of  $T_{\text{aut}}$ .

3)  $K_*(T)$  is a subclass of  $K_{\text{aut}}^{\text{ec}}(T)$  such that  $M \cong N \in K_* \Rightarrow M \in K_*$  and if  $M \subseteq N$  are from  $K_{\text{aut}}^{\text{ec}}$  then  $M \in K_* \Leftrightarrow N \in K_*$ ; there are  $\leq 2^{|T|}$  such classes.

4)  $K_*$  is cute, etc.

5)  $\mathfrak{C}_{\text{aut}}$  is a monster model for  $K_{\text{aut}}^{\text{ec}}$ , i.e., a member of  $K_{\text{aut}}^{\text{ec}}$  which is  $\bar{\kappa}$ -saturated of cardinality  $\bar{\kappa}$ ; it is unique if  $K_{\text{aut}}(T)$  has the JEP.

6) A class  $K_*$  is stable<sup>1</sup> if for some  $\lambda < \bar{\kappa}$  there is no model  $M \in K_*$ ,  $m < \omega$ ,  $\bar{a}_i \in {}^m M$ ,  $i < \lambda$  and q.f. formula  $\varphi(\bar{x}, \bar{y})$  which order  $\{\bar{a}_i : i < \lambda\}$ .

7)  $K_*$  is simple if there is a q.f. formula  $\varphi(\bar{x}, \bar{y})$  and  $m$  such that for every  $\lambda, \kappa$  we can find  $M \in K_*$ ,  $\bar{a}_\eta \in {}^{\ell g(\bar{y})} M$  for  $\eta \in {}^\kappa \lambda$  and  $\bar{b}_\nu \in {}^{\ell g(\bar{x})} M$  for  $\nu \in {}^\kappa \lambda$  such that:

(i)  $M \models \varphi(\bar{b}_\eta, \bar{a}_{\eta \restriction \alpha})$  for  $\alpha < \kappa, \eta \in {}^\kappa \lambda$

(ii) no sequence in  $m$  realizes  $\geq m$  of the formulas  $\langle \varphi(\bar{x}, \bar{a})_{\eta \restriction \alpha} : i < \lambda \rangle$ .

On such models see [Sh 54], [xx], [xx].

**1.4 Fact:** If  $T_{\text{aut}}^{\text{mc}}$  exists then  $K_{\text{aut}}^{\text{ec}}(T)$  is the class of its models.

**1.5 Claim.** In the claims below we can replace “ $T$  has model completion” by dealing with the class  $K_{\text{aut}}^{\text{ec}}(T)$ , and replace  $T^*$  is a model completion by dealing with  $K_*$ .

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<sup>1</sup>this is for classes as above, for general non first order classes this does not fit

**1.6 Claim.** *Let  $T$  be stable,  $T_{\text{aut}}^{\text{mc}}$  exists. The (a)  $\Leftrightarrow$  (b) where*

- (a)  $T_{\text{aut}}^{\text{mc}}$  is stable (i.e., every completion is stable)
- (b) if  $M_0 \prec M_\ell \prec \mathfrak{C}$  for  $\ell = 1, 2$  and  $M_1 \amalg_{M_0} M_2$  then in  $\mathfrak{C}^{\text{eq}}$ ,  $\text{acl}_{\mathfrak{C}^{\text{eq}}}(M_1 \cup M_2) = \text{dcl}_{\mathfrak{C}^{\text{eq}}}(M_1 \cup M_2)$
- (c)  $T_{\text{aut}}^{\text{mc}}$  is dependent (i.e., every completion does not have the independence property).

*Proof.* (b)  $\Rightarrow$  (a)

We work in  $\mathfrak{C}^{\text{eq}}$  and use observation 1.7 below. Suppose  $\mathfrak{C}_* = (\mathfrak{C}, \sigma_*)$  is an expansion of  $\mathfrak{C}^{\text{eq}}$  to a model of  $T_{\text{aut}}^{\text{mc}}$  and let  $\sigma_*^{\text{eq}}$  be the unique extension of  $\sigma_*$  to an automorphism of  $\mathfrak{C}^{\text{eq}}$ . Let  $\lambda = \lambda^{|T|}$ ,  $M^+ \prec (\mathfrak{C}^{\text{eq}}, \sigma_*^{\text{eq}})$ ,  $|M^+| = \lambda$  (note  $|T| \geq \aleph_0$  here (by 1.1(1))).

Now for every  $p \in \mathcal{S}(M^+, \mathfrak{C}_*)$  let  $a_p \in \mathfrak{C}$  realize  $p$  in  $(\mathfrak{C}, \sigma_*)$  and let  $M_p^+, N_p^+$  be such that

$$M_p^+ \prec M^+, \|M_p^+\| = |T| + \aleph_0$$

$$M_p^+ \prec N_p^+ \prec \mathfrak{C}_\sigma, \|N_p^+\| = |T|$$

$$a_p \in N_p^+$$

$$N_p^+ \restriction \tau_T \amalg_{M_p^+ \restriction \tau_T}^{\mathfrak{C}} M^+ \restriction \tau_T.$$

Let  $A_p = \text{acl}_{\mathfrak{C}^{\text{eq}}}(|N_p^+| \cup |M_p^+|)$ . We define a two-place relation  $E$  on  $\mathcal{S}(M^+, \mathfrak{C}_\sigma)$  as follows:

- $\otimes$   $pEq$  iff  $M_p^+ = M_q^+$  and there is an isomorphism  $f$  from  $N_p^+$  onto  $N_q^+$  which is the identity on  $M_p^+$  and satisfying  $f_p(a_p) = a_q$ .

Clearly

$$\otimes_0 \quad E \text{ is an equivalence relation on } \mathcal{S}(M^+, \mathfrak{C}_*)$$

$$\otimes_1 \quad |\mathcal{S}(M^+, \mathfrak{C}_*)/E| \leq \lambda^{|T|}.$$

Hence it is enough to prove that

$$\otimes_2 \quad pEq \Rightarrow p = q.$$

*Proof of  $\otimes_2$ .* Let  $f$  witness  $pEq$ ,

Let  $f^+ : A_p = \text{dcl}_{\mathfrak{C}^{\text{eq}}}(|N_p^+| \cup |M^+|) \rightarrow A_q$  extends  $f \cup \text{id}_M$  and be an elementary mapping (in  $\mathfrak{C}^{\text{eq}}$ ); by non forking calculus it exists and is unique. Obviously it commutes with  $\sigma_*$ . Also  $A_p$  (and  $A_q$ ) are algebraically closed sets in  $\mathfrak{C}^{\text{eq}}$  by our hypothesis (that is, clause (b)) applied to  $|M_p^+|, |N_p^+|, |M^+|$  hence by 1.7(4), 1.8(4) below,  $f^+$  can be extended to an automorphism of  $\mathfrak{C}^{\text{eq}}$ . So by properties of model completion (and the obvious 1.8(1) below) we are done.

$\neg(b) \Rightarrow \neg(a)$ :

Let  $M_0, M_1, M_2$  form a counterexample to (b). So let  $e \in \text{acl}_{\mathfrak{C}^{\text{eq}}}(M_1 \cup M_2) \setminus \text{dcl}_{\mathfrak{C}^{\text{eq}}}(M_1 \cup M_2)$  hence we can find  $\bar{a} \in {}^{\omega>}(M_1), \bar{b} \in {}^{\omega>}(M_2)$  and  $n < \omega, \varphi(x, \bar{b}, \bar{a})$  such that

- $\otimes(i) \quad \mathfrak{C}^{\text{eq}} \models \varphi[e, \bar{b}, \bar{a}]$
- $(ii) \quad \models (\exists^{!n} x) \varphi(x, \bar{b}, \bar{a})$
- $(iii) \quad n \text{ minimal under } (i) + (ii).$

We know  $\varphi(x, \bar{b}, \bar{a}) \vdash \text{tp}(e, M_1 \cup M_2)$  and let  $\{e_0, \dots, e_{n-1}\}$  list  $\varphi(\mathfrak{C}^{\text{eq}}, \bar{b}, \bar{a})$ .

Let  $\bar{e} = \langle e_0, \dots, e_{n-1} \rangle$ . Possibly increasing  $\bar{a}, \bar{b}$  for some formula  $\psi = \psi(\bar{x}, \bar{b}, \bar{a})$  with  $\bar{x} = \langle x_\ell : \ell < n \rangle$  we have  $\models \psi(\bar{e}, \bar{b}, \bar{a})$  and  $\psi(\bar{x}, \bar{b}, \bar{a}) \vdash \text{tp}(\bar{e}, M_1 \cup M_2)$ . So we can find  $f$  such that

- $\otimes \quad f$  is an elementary mapping in  $\mathfrak{C}$
- $\text{Dom}(f) = M_1 \cup M_2 \cup \bar{e}$
- $f \upharpoonright (M_1 \cup M_2)$  is the identity
- $f(e_0) \neq e_0$  (but of course  $f$  permutes  $\{e_\ell : \ell < n-1\}$ ).

Let  $f(\bar{e}) = \bar{e}'$ . Let  $\bar{e}_0 = \bar{e}, \bar{e}_1 = f(\bar{e})$ .

We can find a sequence of  $\mathfrak{C}^{\text{eq}}$ -elementary mapping  $\langle g_i : i < |T|^+ \rangle$  such that

$$\text{Dom}(g_i) = \text{acl}_{\mathfrak{C}^{\text{eq}}}(M_1 \cup M_2)$$

$$g_i \upharpoonright M_2^{\text{eq}} = \text{id}$$

$$\bigcup_{M_2^{\text{eq}}} \{\text{Rang}(g_i) : i < |T|^+\}.$$

Now

- ⊗ if  $k < \omega, i_0 < \dots < i_{k-1} < \omega$  and  $\eta \in {}^{n_2}2$  then the type  $p_\eta = \text{tp}(g_{i_0}(\bar{e}_{\eta(0)}) \hat{\ } g_{i_1}(\bar{e}_{\eta(1)}) \hat{\ } \dots g_{i_{k-1}}(\bar{e}_{\eta(k)}), \bigcup_{i < |T|} \text{Rang}(g_i))$  does not depend on  $\eta$ .

[Why? By induction on  $k$ , hence by transitivity of equality it is enough to prove  $p_\eta = p_\nu$  when  $1 = |\{\ell : \eta(\ell) \neq \nu(\ell)\}|$ .

By an indiscernible sequence = indiscernible set (= symmetry of nonforking, etc.) without loss of generality  $\eta(0) \neq \nu(0)$ . As  $\text{Rang}(\bar{e}_0) = \text{Rang}(\bar{e})$ , without loss of generality  $\bigwedge_{\ell < k-1} \eta(1+\ell) = 0 = \nu(1+\ell)$ . Lastly,  $\text{tp}(\bigcup_{i > 0} \text{Rang}(g_i), \text{Rang}(g_0))$  is finitely satisfiable in  $M_2$  so by the choice of  $\psi$  we are done.]

Now for any  $\eta \in (|T|^+)^2$  we define the function  $h_\eta$ :

$$\text{Dom}(h_\eta) = M_2^{\text{eq}} \cup \{g_i''(M_1^{\text{eq}}) : i < |T|^+\} \cup \{g_i(\bar{e}) : i < |T|^+\}$$

$$h_\eta \upharpoonright M_2^{\text{eq}} = \text{identity}$$

$$h_\eta \upharpoonright g_i''(M_1^{\text{eq}}) = \text{identity}$$

$$h_\eta(g_i(\bar{e})) = \begin{cases} g_i(\bar{e}) = g_i(\bar{e}_0) & \text{if } \eta(i) = 0 \\ g_i(\bar{e}_1) & \text{if } \eta(i) = 1 \end{cases}$$

We can find  $M_3, M_4, \sigma$  such that

$$\cup \{g_i(M_1) : i < |T|^+\} \subseteq M_3 \prec M_4 \prec \mathfrak{C}$$

$$M_2 \bigcup_{M_0} M_4$$

$$M_4 \text{ is saturated of cardinality } > \|M_3\|$$

$$\sigma \in \text{Aut}(M_4), \sigma \upharpoonright M_3 = \text{identity}$$

$$(M_4, \sigma) \text{ is a model of } T_{\text{aut}}^{\text{mc}}.$$

Now for every  $\eta \in (|T|^+)^2$  we can find  $(M_\eta^5, \sigma) \models T_{\text{aut}}$  such that  $(M_4, \sigma) \subseteq (M_\eta^5, \sigma)$  and  $\bar{b}_\eta$  realizing  $\text{tp}_{\mathfrak{C}^{\text{eq}}}(\bar{b}, M_0, \mathfrak{C})$  such that



$$\eta(i) = 0 \Leftrightarrow (\exists \bar{x})(\psi(\bar{x}, \bar{b}_\eta, g_i(\bar{a})) \wedge \sigma(\bar{x}) = x).$$

So  $\{(\exists \bar{x})(\psi(\bar{x}, \bar{y}, g_i(\bar{a})) : i < |T|^+\}$  is an independent set of formulas in  $(M_4, \sigma)$  hence  $T_{\text{aut}}^{\text{mc}}$  is unstable.

$(a) \Rightarrow (d)$ :

Trivial.

$\neg(b) \Rightarrow \neg(c)$ :

Included in the proof of  $\neg(b) \Rightarrow \neg(a)$ .

$\square_{1.6}$

*1.7 Observation.* Assume  $T_{\text{aut}}^{\text{mc}}$  exists,  $T_*$  any completion of it.

- 1) If  $\mathfrak{C}$  is a saturated model of  $T$  of cardinality  $\bar{\kappa} = \bar{\kappa}^{<\bar{\kappa}}$ , can be expanded to a model  $\mathfrak{C}_*$  of  $T_*$ .
- 2) If  $M \models T, \sigma \in \text{Aut}(M)$ , let  $\sigma^{\text{eq}}$  be the natural extension of  $\sigma$  to an automorphism of  $M^{\text{eq}}$ , then (it exists and is unique)  $(M^{\text{eq}}, \sigma^{\text{eq}}) \models (T^{\text{eq}})_{\text{aut}}$ .
- 3)  $(T^{\text{eq}})_{\text{aut}}$  has a model completion  $T$  and there is a natural one to one correspondence between the completions of the model completions of  $(T^{\text{eq}})_{\text{aut}}$  and  $\{T_{**} : T_{**} \text{ a model completion of } T_{\text{aut}}^{\text{mc}}\}$  any one of the former is essentially bi-interpretable with the corresponding one of the latter (but we have the elements not in any  $P_{E(\bar{x}, \bar{y})}$ ).
- 4) Let  $\mathfrak{C}_* = (\mathfrak{C}, \sigma_*)$  be a  $\bar{\kappa}$ -saturated model of  $T_*$  expanding  $\mathfrak{C}$ . If  $A_\ell \subseteq \mathfrak{C}^{\text{eq}}, A_\ell = \text{acl}_{\mathfrak{C}^{\text{eq}}}(A_\ell), A_\ell$  closed under  $\sigma_*, f$  is an  $\mathfrak{C}^{\text{eq}}$ -elementary mapping from  $A_1$  onto  $A_2$  commuting with  $\sigma$  then  $f$  can be extended to an automorphism of  $(\mathfrak{C}^{\text{eq}})_{\text{aut}}$  (it is  $\mathfrak{C}^{\text{eq}}$  expanded by  $\sigma$  naturally extended to  $\sigma^+$ ).

*1.8 Observation.* 1)  $M$  is a model of  $T, \sigma_* \in \text{Aut}(M)$  iff  $(M, \sigma_*)$  is a model of  $T_{\text{aut}}$ .

2) If  $M \prec \mathfrak{C}$  and  $(M, \sigma_*)$  as a model of  $T_{\text{aut}}$  then for one and only one  $\sigma_*^{\text{eq}} \in \text{Aut}(M^{\text{eq}})$  extend  $\sigma_*$ .

3) If  $M \prec \mathfrak{C}, \sigma_*^{\text{eq}} \in \text{Aut}(M^{\text{eq}})$  then  $\sigma_*^{\text{eq}} \upharpoonright M \in \text{Aut}(M)$ .

4) If  $A_\ell \subseteq \mathfrak{C}^{\text{eq}}$  and  $A_0 = \text{acl}_{\mathfrak{C}^{\text{eq}}}(A_0)$  and  $f_\ell$  is an  $\mathfrak{C}^{\text{eq}}$ -elementary mapping from  $A_\ell$  onto  $A_\ell$  for  $\ell = 0, 1, 2$  and  $f_0 \subseteq f_1, f_0 \subseteq f_2$  then for some automorphism  $F$  of  $\mathfrak{C}^{\text{eq}}, F \upharpoonright A_0 = \text{id}_{A_0}$  and  $f_2 \cup (F f_1 F^{-1})$  is an elementary mapping in  $\mathfrak{C}^{\text{eq}}$  (hence can be extended to an automorphism of  $\mathfrak{C}^{\text{eq}}$ ; if  $A_1 \cup A_2$  then without loss of generality  $F \upharpoonright$

$A_0$

$(A_1 \cup A_2) = \text{id}_{A_1 \cup A_2}$ .

## §2

2.1 Example: There is  $T$  such that:

- (a)  $T$  is as in 1.1, stable  $T_{\text{aut}}^{\text{mc}}$  exists. Moreover  $T$  is superstable, countable  
 $I(\aleph_\alpha, T) \leq 2^{|\alpha|}$  for  $\alpha \geq 2^{\aleph_0}$  (hence NDOP, NOTOP, shallow with small  
depths, with  $\leq 2^{\aleph_0}$  dimensions)
- (b)  $T_{\text{aut}}^{\text{mc}}$  exist
- (c) some completions of  $T_{\text{aut}}^{\text{mc}}$  are stable and some are not.

*Proof.* Let us define  $M, I$

- $|M|$  is  $\{(\eta, k, n, \ell) : k, n < \omega, \ell < 2 \text{ and } \eta \in {}^\omega 2\}$  and  $k = n \Rightarrow \ell = 0$
- $E_n^M$ , a two-place relation is  $\{(\eta_1, k_1, n_1, \ell_1), (\eta_2, k_2, n_2, \ell_2) \in |M| \times |M| : \eta_1 \upharpoonright n = \eta_2 \upharpoonright n\}$
- $E^M$ , a two-place relation is  $\{(\eta_1, k_1, n_1, \ell_1), (\eta_2, k_2, n_2, \ell_2) \in |M| \times |M| : \eta_1 = \eta_2\}$
- $Q^M$ , a one-place relation is  $\{(\eta, k, n, \ell) \in |M| : k = n\}$
- $F_1^M$ , a one-place relation is:  $F_1^M((\eta, k, n, \ell)) = (\eta, k, k, 0)$
- $F_2^M$ , a one-place relation is:  $F_2^M((\eta, k, n, \ell)) = (\eta, n, n, 0)$

Let  $T = \text{Th}(M)$ . Clearly it satisfies (a):

- ⊗<sub>1</sub>  $T_{\text{aut}}^{\text{mc}}$  exists.  
[Why? Check that there are no obstructions.]
- ⊗<sub>2</sub>  $T_{\text{aut}}^{\text{mc}}$  has an unstable completion.  
[Why? By 1.6, or more specifically, see below.]

We shall now prove

- ⊗<sub>2</sub><sup>+</sup> for  $T_*$  a completion of  $T_{\text{aut}}^{\text{mc}}$ ,  $T_*$  is unstable if:  
for some  $M^+ \models T_*$ , for some  $a \in M^+$  we have  $\bigwedge_n a E_n(\sigma^{M^+}(a))$  or just  
 $(\exists m) \bigwedge_{n < \omega} a E_n((\sigma^{M^+})^m(a))$ , i.e., for some  $m^* \in [1, \omega)$  we have  $\bigwedge_n a E_n a_{M^+}$   
where  $a_0 = a, a_{\ell+1} = \sigma^{M^+}(a_\ell)$  for  $\ell < \omega$ .

Let  $m^*, a, \langle a_\ell : \ell < \omega \rangle$  be as above. We define  $N$  a model of  $T$ : let  $|N|$ , the universe of  $N$  be

$$|M^+| \cup \{(m, k, n, \ell) : m < m^*, k, n < \omega, \ell < 2, k = n \Rightarrow \ell = 0\}$$

we assume no incidental identification.

$$E_n^N : \begin{cases} E_n^N \text{ is an equivalence relation} \\ E_n^N \upharpoonright |M^+| = E_n^{M^+} \\ \text{every } (m, k, n, \ell) \in |N| \setminus |M^+| \text{ is } E_n\text{-equivalent to } a_m \end{cases}$$

$$E^N : \begin{cases} E^N \text{ is an equivalence relation} \\ E^N \upharpoonright |M^+| = E^N \\ \{(m, k, n, \ell) \in |N| \setminus |M^+| : k, n < \omega, \ell < 2, k = n \Rightarrow \ell = 0\} \\ \text{is an } E^N\text{-equivalence class (for each } m < m^*) \end{cases}$$

$$Q^N = Q^N \cup \{(m, k, k, 0) : k < \omega\}$$

$$F_1^N \text{ extends } F_1^{M^+}, F_1^N((m, k, n, \ell)) = (m, k, k, 0)$$

$$F_2^N \text{ extends } F_2^{M^+}, F_2^N((m, k, n, \ell)) = (m, n, n, 0).$$

Easily

$$\Box_1 \quad M^+ \upharpoonright \tau_T \prec N.$$

Now we define an automorphism  $\sigma^+$  of  $N$ :

$$\Box_2 \quad \sigma^+ \upharpoonright |M^+| = \sigma^{M^+}$$

$$\Box_3 \quad \text{if } m_1, m_2 < m^*, m_2 = m_1 + 1 \bmod m^* \text{ then}$$

$$\sigma(m_1, n, k, \ell) \text{ is:}$$

$$(m_2, n, k, 1 - \ell) \text{ if } m_1 = m^* - 1 \text{ \& } n < k$$

$$(m_2, n, k, \ell) \text{ otherwise.}$$

Easy to check that  $\sigma^+ \in \text{Aut}(N)$ , so  $(N, \sigma) \supseteq M^+$  is a model of  $T_{\text{aut}}$ . As  $T_{\text{aut}}^{\text{mc}}$  exists and  $M^+ \models T_{\text{aut}}^{\text{mc}}$  there is a model  $N^+ \models T_{\text{aut}}^{\text{mc}}$  such that  $M^+ \prec M^+, (N, \sigma) \subseteq N^+$ .

Let

$$\varphi(x, y) = Q(x) \ \& \ Q(y) \ \& \ xEy \ \& \ (\exists z)(F_1(z) \ \& \ F_2(z) = y \ \& \ (\sigma^{m^*}(z) \neq z))$$

This is a first order formula in  $\mathbb{L}(\tau_{\text{Th}(M^+)}) = \mathbb{L}(\tau_{T_{\text{aut}}})$  and  $N^+ \models \varphi[b_n, b_k]$  iff  $n < \omega$  where  $b_n = (0, n, n, 0) \in N \subseteq N^+$ , so this formula has the order property in  $\text{Th}(N^+) = \text{Th}(M^+)$ . So  $\text{Th}(M^+)$  is unstable as required in  $\otimes_2^+$

⊗<sub>3</sub> if  $T_*$  is a completion of  $T_{\text{aut}}^{\text{mc}}$  not satisfying the demand in ⊗<sub>2</sub><sup>+</sup> then  $T_*$  is stable.

[Why? As any model  $M^+$  of  $T_*$ ,  $\sigma^{M^+}$  acts as a permutation of  $|M^+|/E^{M^+}$  which has no fix point and even no finite cycle. Now reflect.]

⊗<sub>4</sub> there is a completion  $T_*$  of  $T_{\text{aut}}^{\text{mc}}$  which is stable.

Why? Let  $f$  be a permutation of  ${}^\omega 2$  such that

$$(\alpha) \quad \eta, \nu \in {}^\omega 2 \wedge \eta \upharpoonright n = \nu \upharpoonright n \Rightarrow f(\eta) \upharpoonright n = f(\nu) \upharpoonright n$$

(β) for every  $m < \omega$  ( $\geq 2$ ) for some  $n < \omega$  we have if  $\eta \in {}^\omega 2$ , then  $\eta, f^m(\eta)$  are not  $E_n$ -equivalent.

Easy to construct (or use  $\prod_{n < \omega} (n+1)$  instead  ${}^\omega 2$ ) and define  $M^+$ , a  $\tau_{T_{\text{aut}}}$ -expansion of  $M$  by defining

$$\sigma^{M^+}((\eta, k, n, \ell)) = (f(\eta), k, n, \ell).$$

So if  $M^+ \subseteq N^+ \models T_{\text{aut}}^{\text{mc}}$  then  $T_* = \text{Th}(N^+)$  fail the demand in ⊗<sub>2</sub><sup>+</sup> hence by ⊗<sub>3</sub> it is stable as required (and it is uniquely determined by  $M^+$ , really just the action on  $\text{acl}_{\mathfrak{C}^{eq}}(\emptyset)$ , suffice. So ⊗<sub>4</sub> holds.  $\square_{2.1}$

**2.2 Discussion:** It seems reasonable that we can characterize when this occurs thus answering fully 0.1; see below.

A closely related example is

**2.3 Claim.** *There is  $T$  such that:*

- (a)  $T$  is stable (complete countable first order theory) and has elimination of quantifiers for simplicity
- (b)  $T$  is superstable and small, i.e., with countable  $D(T)$
- (c)  $T_{\text{aut}}$  has no model completion
- (d) some  $T_{\text{aut}}(M^+)$  has a model completion where

**2.4 Definition.** 1) For a model  $M^+ = (M, \sigma^{M^+})$  of  $T_{\text{Aut}}$  let  $T_{\text{aut}}(M^+) = T_{\text{aut}} \cup \text{Th}(M, c)_{c \in M} \cup \{\sigma(c_1) = c_2 : \sigma^{M^+}(c_1) = c_1\}$ .

**2.5 Remark.** Actually we can use any completion of  $T_{\text{aut}} \cup$  (the action of  $\sigma$  on  $\text{acl}_{\mathfrak{C}^{eq}}(\emptyset, \mathfrak{C}_T)$  (i.e., on the  $E$ -equivalence classes for each  $n$ ).

*Proof.* Define  $M$

- (a)  $\tau_M = \{E_n, P_n : n < \omega\} \cup \{E, E_*\}$
- (b)  $|M| = \{(\eta, k, n, \ell) : \eta \in {}^\omega 2, k < \omega, n < \omega, \ell < 2\}$
- (c)  $E_n^M = \{(\eta_1, k_1, n_1, \ell_1), (\eta_2, k_2, \eta_2, \ell_2)) \in |M| \times |M| : \eta_1 \upharpoonright n = \eta_2 \upharpoonright n\}$
- (d)  $E^M = \{(\eta_1, k_1, n_1, \ell_1), (\eta_2, k_2, \eta_2, \ell_2)) \in |M| \times |M| : \eta_1 = \eta_2 \text{ and } k_1 = k_2\}$
- (e)  $E_*^M = \{((\eta_1, k_1, n_1, \ell_1), (\eta_2, k_2, \eta_2, \ell_2)) \in |M| \times |M| : \eta_1 = \eta_2, k_1 = k_2, n_1 = n_2\}$
- (f)  $P_n^M = \{(\eta, k, n, \ell) \in |M| : n = m\}$ .

We choose  $\sigma^M$  such that  $\sigma(\eta, k, n, \ell) = (\eta', k, n, \ell)$  and  $(\eta, \eta')$  are as in the proof of 2.1.

*Remark.* If we let  $(d)'$  be as in 2.8 below we add  $\sigma = \text{the identity}$  then  $(a) + (c) + (d)'$  is impossible by [BlSh 759].

Actually the case  $\sigma$  is the identity on some  $M$  is the real one because

**2.6 Claim.** *For any first order complete  $T_1$  (with  $\tau_{T_1}$  a set of predicates for simplicity) there is  $T$  such that:*

- (a)  $T$  is first order complete
- (b) if  $a \in M, M \models T$  then we can interpret  $T_1$  in  $(M, a)$
- (c)  $\tau_T \setminus \tau_{T_1}$  countable
- (d) some  $T_{\text{aut}}(M^+)$  has a model completion.

*Proof.* As in 2.3 without  $E_*, P_n(n < \omega)$  in any  $E^M$ -equivalence class we “plant” a model of  $T_1$ .

**2.7 Claim.** *Let  $T_*$  be a completion of  $T_{\text{aut}}^{\text{mc}}$ .*

*The following are equivalent:*

Condition (a):  $T_*$  is stable.

Condition (b): If  $T$  is stable and  $(\alpha) + (\beta) + (\gamma)$  below holds, then  $(*)$  below holds where

- ( $\alpha$ )  $M_0^+ \prec M_\ell^+ < M_3^+$  for  $\ell = 1, 2, M_0 \models T_*, M_\ell \models T_{\text{aut}}$  for  $\ell = 1, 2, 3$  and
- ( $\beta$ )  $M_\ell = M_\ell \upharpoonright \tau_T$  and  $M_1 \bigcup_{M_0}^{M_3} M_2$  without loss of generality  $M_3 \prec \mathfrak{C} = \mathfrak{C}_T$

- ( $\gamma$ ) if  $f$  is an elementary mapping from  $\text{acl}_{\mathfrak{C}^{\text{eq}}}(M_1 \cup M_2)$  onto itself extending  $\sigma^{M_1^+} \cup \sigma^{M_2^+}$
- (\*) there is an elementary mapping  $h$  from  $\text{acl}_{\mathfrak{C}}(M_1 \cup M_2)$  onto itself such that  $h \upharpoonright (M_1 \cup M_2) = \text{identity}_{M_1 \cup M_2}$  and  $hfh^{-1} = \sigma^{M_3^+} \upharpoonright \text{acl}_{\mathfrak{C}^{\text{eq}}}(M_1 \cup M_2)$ .

*Proof.* (b)  $\Rightarrow$  (a):

As in the proof of 1.6.

$\neg(b) \Rightarrow \neg(a)$ :

We can use compactness to replace  $\neg(b)$  by a finite failure, and continue as in the proof of 1.6.

*2.8 Remark.* We can make  $\neg(b)$  more explicit as in the proof of 2.7.

§3 NSOP<sub>3</sub>

As by [KkSh 748], if  $T_{\text{aut}}^{\text{mc}}$  exists, then  $T$  fails the strict order property. It seems reasonable to ask if any  $T_{\text{aut}}^{\text{mc}}$ , which exists, can have the strict order property. As we understand the stable case, it seems reasonable to deal with it. In fact, more turn out to hold.

**3.1 Claim.** *[ $T$  as in 1.1.] If  $T$  is stable, any completion  $T_*$  of  $T_{\text{aut}}^{\text{mc}}$  satisfies NSOP<sub>3</sub> (see [Sh 500, §2] and [ShUs 789]).*

*Proof.* 1) Clause (a):

Let  $T_*$  be completion of  $T_{\text{aut}}^{\text{mc}}$  and  $\varphi(\bar{x}, \bar{y})(\ell g(\bar{x}) = \ell g(\bar{y}) = n^* < \omega)$  a first order formula in  $\mathbb{L}(\tau_{T_*})$  and for some  $M \models T_*$  we have  $M \models \varphi(\bar{a}_n, \bar{a}_m)^{\text{if}(n < m)}$ . Hence we can find an E.M.-template  $\Phi$  such that  $\tau_\Phi \supseteq \tau_{T_*} = \tau_T \cup \{\sigma\}$  and for linear orders  $I \subseteq J$ ,  $\text{EM}(I, \Phi) \prec \text{EM}(J, \Phi) \neq T_*$ , with skeleton  $\langle \bar{a}_t : t \in J \rangle$  such that  $\text{EM}(J, \Phi) \models \varphi[\bar{a}_s, \bar{a}_t]^{\text{if}(s < t)}$  for  $s, t \in J$  (so  $\bar{a}_t \in \text{EM}(\{t\}, \Phi)$  (see, e.g., [Sh:c, VII] or [Sh:e, III]). Now (recalling that  $\text{EM}_\tau(I, \Phi) = \text{EM}(I, \Phi) \upharpoonright \tau$ ) without loss of generality

- ⊗<sub>1</sub> if  $I_1, I_2 \subseteq J, I_0 = I_1 \cap I_2$  and if  $t \in I_1 \setminus I_0$  then there is  $s \in I_0$  such that  $s < t$  &  $[s, t]_{J \cap I_2} \subseteq I_0$  or  $t < s$  &  $[t, s]_{J \cap I_2} \subseteq I_0$  then  $\text{tp}_{L(\tau_{T_*})}(\text{EM}_{\tau_{T_*}}(I_1, \Phi), \text{EM}_{\tau_{T_*}}(I_2, \Phi))$  is f.s. (finitely satisfiable) in  $\text{EM}_{\tau_{T_*}}(I_0, \Phi)$   
 [Why? Let  $I \times \mathbb{Z}$  be ordered lexicographically, choose  $\Phi'$  such that  $\text{EM}(I, \Phi') = \text{EM}(I \times \mathbb{Z}, \Phi)$ , with skeleton  $\bar{a}'_t = \bar{a}_{(t, 0)}$ ; can look at [Sh 394].]

For  $u \subseteq \{0, 1, 2\}$  let  $M_u^2 = \text{EM}(u, \Phi)$  and if  $|u| = |v|$  both subsets of  $\{0, 1, 2\}$  let  $f_{v,u}$  be the canonical isomorphism from  $M_u$  onto  $M_v$ . Let  $M_u^1 = M_u^2 \upharpoonright \tau_{T_*}, M_u^0 = M_u^2 \upharpoonright \tau_T$ . Let  $N$  be such that  $M_{\{0,1,2\}}^0 \prec N, N$  is  $\|M_{\{0,1,2\}}^0\|^+$ -saturated

- ⊗<sub>2</sub> in  $N, \bigcup_{M_\emptyset^0} \{M_{\{0\}}^0, M_{\{1\}}^0, M_{\{2\}}^0\}$   
 [Why? By ⊗<sub>1</sub> and nonforking calculus.]

Let  $g_0 =: f_{\{0\}, \{2\}} \cup f_{\{2\}, \{0\}}$

- ⊗<sub>3</sub>  $g_0$  is an elementary mapping (inside  $N$ )  
 [Why? Nonforking calculus.]

Let  $g_1$  be an elementary mapping inside  $N$  extending  $g_0$  with domain  $M_{\{0,2\}}^0$ .

Let  $M_{\{0,2\}}^{0,*} = g(M_{\{0,1\}}^0)$ .

Let  $M_{\{0,2\}}^{1,*}$  be an expansion of  $M_{\{0,2\}}^{0,*}$  by an automorphism  $\sigma^{M_{\{0,2\}}^{1,*}}$  such that  $g_1$  is an isomorphism from  $M_{\{0,2\}}^1$  onto  $M_{\{0,2\}}^{1,*}$ , clearly exists.

As  $N$  is a model of the stable theory  $T$  without loss of generality  $\text{tp}_{L^*(\tau_T)}(|M_{\{0,2\}}^{1,*}|, |M_{\{0,1,2\}}^0|)$  does not fork over  $|M_{\{0\}}^0| \cup |M_{\{2\}}^0|$ .

Now the point is that

- ⊙  $h = \sigma^{M_{\{0,1\}}^1} \cup \sigma^{M_{\{0,2\}}^{1,*}} \cup \sigma^{M_{\{1,2\}}^1}$  is a permutation of  $|M_{\{0,1\}}^{1,*}| \cup |M_{\{0,1\}}^1| \cup |M_{\{1,2\}}^1|$  and is an elementary mapping.  
 [Why? Let  $B_0 = |M_{\{0\}}^0| \cup |M_{\{2\}}^0|$ ,  $B_1 = |M_{\{0,1\}}^0| \cup |M_{\{2,2\}}^0|$ .  
 By [Sh:c, XII], the pair  $(B_0, B_1)$  satisfies the T.V. condition inside  $N$  (i.e., if  $\varphi(\bar{x}, \bar{y}) \in \mathbb{L}(\tau_T)$ ,  $N \models \varphi[\bar{a}, \bar{b}]$ ,  $\bar{a} \subseteq B_1$ ,  $\bar{b} \subseteq B_0$  then for some  $\bar{a}' \subseteq B_0$ ,  $N \models \varphi[\bar{a}', \bar{b}]$ . Moreover, we can allow  $\bar{b} \subseteq |M_{\{0,2\}}^{0,*}|$  then this follows.]

So for some  $N', N \prec N' \models T$  and there is an automorphism  $h'$  of  $N'$  extending  $h$  and we can extend  $(N', h')$  to a model  $(N'', h'')$  of  $T_*$ . By this model clearly

$$(N'', h'') \models \varphi[\bar{a}_0, a_1] \text{ using } M_{\{0,1\}}^1$$

$$(N'', h'') \models \varphi[\bar{a}_1, \bar{a}_2] \text{ using } M_{\{1,2\}}^1$$

$$(N'', h'') \models \varphi[\bar{a}_2, \bar{a}_0] \text{ using } M_{\{0,2\}}^{1,*} \text{ and}$$

$$g_1 \text{ being an isomorphism from } M_{\{0,2\}}^1 \text{ onto } M_{\{0,2\}}^{1,*}.$$

This is enough to show  $T_* \models \text{NDOP}_3$ .

**3.2 Claim.**  *$T$  is stable or just simple then any  $T_*$  (assuming it exists,  $K_*$  in general) is simple.*

*Proof.* We write it for  $K_*$ . Choose  $\kappa = \text{cf}(\kappa) > |T|$  and  $\mu$  a strong limit singular cardinal of cofinality  $\kappa$ . Let  $\langle \lambda_i : i < \kappa \rangle$  be increasing with limit  $\mu$ ,  $\lambda_0 > \kappa$ ,  $\lambda_\kappa = \mu$ ,  $\langle {}_*(M_i^+ : i < \kappa) \rangle$  be an increasing sequence of elementary submodels of  $\mathfrak{C}_{K_*}$  (check notation),  $\|{}_*(M_i^+)\| = 2^{\lambda_i}$ ,  ${}_*(M_i^+)$  is  $\lambda_i^+$ -homo universal (in  $K_{\text{aut}}^{\text{ec}}(T)$ ),  $M^+ = \cup \{{}_*(M_i^+ : i < \kappa)\}$ . Let  $\langle p_i^+ : i < \mu^+ \rangle$  be a sequence of existential types in  $\mathbb{L}(\tau \cup \{\sigma\})$  each of cardinality  $\leq \kappa$  with domain  $\subseteq M$ , and we shall prove that for some  $\alpha < \beta < \mu^+$ ,  $p_\alpha^+ \cup p_\beta^+$  is realized in  $\mathfrak{C}_{K_*}$ , this suffices.



For each  $\alpha < \mu^+$ , we can find  $a_\alpha \in \mathfrak{C}_{K^*}$  realizing  $p_i$  and  $N_{3,\alpha}^* \prec \mathfrak{C}_{K^*}$  of cardinality  $\kappa$  to which  $a_i$  belongs and  $N_{2,\alpha}^+ = N_{3,\alpha}^+ \cap M^+ \prec M^+$  and  $\text{tp}_{\mathfrak{C}}(|N_{3,\alpha}^+|, |M^+|)$  does not fork over  $|N_{2,\alpha}^+|$ . Let  $N_{1,\alpha}^+ \prec N_{3,\alpha}^+$  be of cardinality  $|T|$  such that  $a_i \in N_{1,\alpha}^+$ ,  $\text{tp}_{\mathfrak{C}}(|N_{1,\alpha}^+|, |M^+|)$  does not fork over  $|N_{0,i}^+|$  where  $N_{0,i}^+ = N_{1,i}^+ \upharpoonright M^+ \prec M^+$ . Without loss of generality  $\alpha < \mu^+ \Rightarrow N_{0,\alpha}^+ = N_0^+$  and for every  $\alpha, \beta < \mu^+$  there is an isomorphism  $h_{\beta,\alpha}$  from  $N_{3,\alpha}^+$  onto  $N_{3,\beta}^+$  mapping  $a_\alpha, N_{1,\alpha}^+, N_{2,\alpha}^+$  to  $a_\beta, N_{1,\beta}^+, N_{2,\beta}^+$  respectively and  $h_{\beta,\alpha} \upharpoonright N_0^+ = \text{id}_{N_0^+}$ . Moreover, without loss of generality for some well ordering  $<^*$  all  $h_{\beta,\alpha}$  are order preserving.

Let  $\kappa > \bar{\kappa}$ ,  $\mathfrak{B}$  be an elementary submodel of  $(\mathcal{H}(\chi), \in)$  of cardinality  $2^\kappa$  such that  $T, \kappa, \mu, \mathfrak{C}, \mathfrak{C}_{K^*}, M^+, \langle N_i^+ : i < \mu^+ \rangle$  belongs and such that  $[\mathfrak{B}] \leq \kappa \subseteq \mathfrak{B}$ . Now choose  $\alpha(2) \in \mu^+ \setminus \mathfrak{B}$ , and let  $M_0^+ = N_{1,\alpha}^+ \upharpoonright \mathfrak{B}$ . Clearly  $M_0^+ \prec M^+$  and there is  $\alpha(1) \in \mu^+ \cap \mathfrak{B}$  such that  $h_{\alpha(1),\alpha(2)}$  is the identity on  $M_0^+$ .

[FILL?]

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